Intertwining of some Markov semigroups on Carnot groups of step two

(Work in progress)

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Outline

Definition

A Carnot group $\mathbb{G}$ of step two is a connected **nilpotent** Lie group such that its Lie algebra $\mathfrak{g}$ has a decomposition of the form

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1], \quad [\mathfrak{g}, \mathfrak{g}_2] = 0.$$

- $\mathbb{G}$ can be identified as $\mathbb{R}^n \times \mathbb{R}^m$ where $n = \dim(\mathfrak{g}_1)$, $m = \dim(\mathfrak{g}_2)$ with the group operation

$$(x, v) \star (x', v') = (x + x', v + v' + (\langle A_l x, x' \rangle)_{l=1}^m), \quad x, x' \in \mathbb{R}^n, v, v' \in \mathbb{R}^m,$$

  where $A_1, \ldots, A_m$ are $n \times n$ skew-symmetric matrices. 0 is identity element.

- $\mathbb{G}$ is unimodular and the Haar measure coincides with the Lebesgue measure on $\mathbb{G} = \mathbb{R}^{n+m}$.

- $\mathbb{G}$ is equipped with dilations $\delta_c \in \mathrm{Aut}(\mathbb{G})$ such that $\delta_c(x, v) = (cx, c^2 v)$ where $x \in \mathbb{R}^n, v \in \mathbb{R}^m$

- Heisenberg group: $\mathbb{H}^{2d+1} = \mathbb{C}^d \times \mathbb{R}$.

$$(z_1, v_1) \star (z_2, v_2) = (z_1 + z_2, v_1 + v_2 + \frac{1}{2} \operatorname{Im}(z_1 \cdot z_2)).$$

- *H*-type groups: $\mathbb{H}_{d,m} = \mathbb{R}^{2d} \times \mathbb{R}^m$

$$(z_1, v_1) \star (z_2, v_2) = (z_1 + z_2, v_1 + v_2 + \frac{1}{2}(\langle U_l z_1, z_2 \rangle)_{l=1}^m),$$

$U_1, \ldots, U_m$ are orthogonal, skew-symmetric, and $U_l U_{l'} = -U_{l'} U_l$ for all $1 \leqslant l, l' \leqslant m$.

- Non-isotropic Heisenberg groups: $\mathbb{H}_\omega^{2d+1} = \mathbb{R}^{2d} \times \mathbb{R}$

$$(z_1, v_1) \star (z_2, v_2) = (z_1 + z_2, v_1 + v_2 + \frac{1}{2}\omega(z_1, z_2)), \quad \omega \text{ is alternating bilinear form.}$$

Outline

On any Carnot group $\mathbb{G}$ with Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, we can consider left-invariant vector fields $\widetilde{X}(g) \in \mathcal{T}_g\mathbb{G}$ defined by

$$\widetilde{X}(g) = (d\tau_g)_1(X), \quad X \in \mathfrak{g}, \tau_g(h) = g \star h,$$

and can define the left-invariant inner product

$$\langle \widetilde{X}(g), \widetilde{Y}(g)\rangle_{\mathcal{T}_g\mathbb{G}} = \langle X, Y\rangle, \quad X, Y \in \mathfrak{g}.$$

Taking an **orthonormal basis** $\{X_1, \dots, X_n\} \in \mathfrak{g}_1$ and $\{Z_1, \dots, Z_m\} \in \mathfrak{g}_2$, we note that

$$(\mathrm{Lie}\{\widetilde{X}_1, \dots, \widetilde{X}_n\}) = \mathcal{T}\mathbb{G}.$$

Thus, $\{\widetilde{X}_1, \dots, \widetilde{X}_n\}$ defines a sub-Riemannian structure on $\mathbb{G}$ with a sub-Laplacian

$$\mathbf{\Delta}_h = \sum_{i=1}^{n} \widetilde{X}_i^2.$$

$\mathbf{\Delta}_h$ is a sub-elliptic and by Hörmander's theorem it is hypoelliptic.

The three dimensional Heisenberg group $\mathbb{H} = \mathbb{R}^3$ is equipped with the group operation

$$(x, y, w) \star (x', y', w') = (x + x', y + y', w + w' + \frac{1}{2}(xy' - x'y)).$$

The left-invariant vector fields in $\mathcal{T}\mathbb{H}$ are generated by

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}$$

- $[X, Y] = Z$, which implies that $\mathcal{H} = \mathrm{Span}\{X, Y\}$ defines a sub-Riemannian structure on $\mathbb{H}$, if X, Y are considered as orthonormal basis of $\mathcal{H}$.

- The sub-Laplacian on $\mathbb{H}$ is defined by

$$\boldsymbol{\Delta}_h = X^2 + Y^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (x^2 + y^2)\frac{\partial^2}{\partial z^2} + \frac{1}{2}\left(y\frac{\partial^2}{\partial x \partial z} - x\frac{\partial^2}{\partial y \partial z}\right)$$

Outline

Objectives

- Obtaining a complete description of the spectrum of $\mathbf{\Delta}_h$ through intertwining with some semigroups on euclidean spaces.

- To see how the spectrum behaves with respect to some Lévy-type perturbations of $\mathbf{\Delta}_h$.

- Proving isospectrality of some Ornstein-Uhlenbeck type operators on $\mathbb{G}$ via intertwining.

Outline

Horizontal heat semigroup on $\mathbb{G}$

Recall the sub-Laplacian/horizontal Laplacian $\mathbf{\Delta}_h = X_1^2 + \cdots + X_n^2$.

$(\mathbf{\Delta}_h, C_c^\infty(\mathbb{G}))$ is essentially self-adjoint in $\mathrm{L}^2(\mathbb{G})$.

Definition

The horizontal heat semigroup $\mathbf{Q}_t$ on $\mathrm{L}^2(\mathbb{G})$ is defined by $\mathbf{Q}_t = e^{t\mathbf{\Delta}_h}$.

- $\mathbf{Q}_t$ is left-translation invariant, that is, $\mathbf{Q}_t(\tau_g \mathrm{f}) = \tau_g \mathbf{Q}_t \mathrm{f}$ where $\tau_g \mathrm{f}(h) = \mathrm{f}(g \star h)$.
- Haar measure on $\mathbb{G}$ is the unique invariant measure (up to multiplication by constants) of $\mathbf{Q}_t$.
- For any $c > 0$ and $t \geqslant 0$, $\delta_c \mathbf{Q}_{tc^2} = \mathbf{Q}_t \delta_c$, where $\delta_c \mathrm{f}(g) = \mathrm{f}(\delta_c g)$.
- $\mathbf{Q}_t$ has smooth transition densities $\mathbf{q}_t(g, h)$ such that

$$\mathbf{Q}_t \mathrm{f}(g) = \int_{\mathbb{G}} \mathbf{q}_t(g, h) \mathrm{f}(h) dh, \quad \forall \mathrm{f} \in \mathrm{L}^1(\mathbb{G}) \cap \mathrm{L}^2(\mathbb{G}).$$

- $\mathbf{q}_t(g, h) = \mathbf{q}_t(0, g^{-1} \star h)$.

H is a Hilbert space and $B : \mathcal{D}(B) \to \mathrm{H}$. B is invertible if B^{-1} is a bounded operator on H.

$$\sigma(B) = \{\lambda \in \mathbb{C} : (\lambda I - B) \text{ is not invertible}\}$$

is a closed subset of $\mathbb{C}$. $\sigma(B)$ can be decomposed into three disjoint parts:

$$\begin{aligned} \textit{point} \quad & \sigma_p(B) = \{\lambda \in \mathbb{C} : \ker(\lambda I - B) \neq \{0\}\} \\ \textit{cont} \quad & \sigma_c(B) = \{\lambda \in \sigma(B) \setminus \sigma_p(B),\ \mathcal{R}(\lambda I - B) \neq \mathrm{H},\ \mathrm{cl}(\mathcal{R}(\lambda I - B)) = \mathrm{H}\} \\ \textit{residual} \quad & \sigma_r(B) = \{\lambda \in \sigma(B) \setminus \sigma_p(B),\ \mathrm{cl}(\mathcal{R}(\lambda I - B)) \subsetneq \mathrm{H}\} \end{aligned}$$

Spectrum contd..

Recall that $\mathbb{G} = \mathbb{R}^{n+m}$.

Theorem

There exists a closed subspace $\mathcal{L} \subset \mathrm{L}^2(\mathbb{G})$ with an orthogonal decomposition

$$\mathcal{L} = \oplus_{n=1}^{\infty} \mathcal{L}_n,$$

and strongly continuous contraction semigroups $(P_t^n)_{t\geqslant 0}$ on $\mathrm{L}^2(\mathbb{R}^{k+m})$ such that

1. $\mathbf{Q}_t \mathcal{L}_n \subseteq \mathcal{L}_n$ *for all* n.
2. $\sigma(P_t^n) = \sigma_c(P_t^n) = [0,1]$ *for all* n.
3. *There exist unitary operators* $U_n : \mathrm{L}^2(\mathbb{R}^{k+m}) \to \mathcal{L}_n$ *such that*

$$\mathbf{Q}_t U_n = U_n P_t^n \quad \text{on } \mathrm{L}^2(\mathbb{R}^{k+m}).$$

We can describe the subspaces $\mathcal{L}, \mathcal{L}_n$ explicitly. It requires some Fourier analysis on $\mathbb{G}$ defined via irreducible unitary representations.

Theorem

For any $t > 0$, $\sigma(\mathbf{Q}_t) = \sigma_c(\mathbf{Q}_t) = [0,1]$.

- When $\mathbb{G}$ is the *H*-type group $\mathbb{H}^{d,m} = \mathbb{R}^{2d} \times \mathbb{R}^m$, then P_t^n is the Poisson semigroup generated by $-(2n+d)\sqrt{-\Delta}$ on $\mathbb{R}^m$.

- In general, P_t^n is **not** Markovian.

Horizontal Brownian motion

The Markov process associated to $\mathbf{Q}_t$ is called the horizontal Brownian motion on $\mathbb{G}$.

Recall that the group operation on $\mathbb{G}$ is given by

$$(x, v) \star (x', v') = (x + x', v + v' + (\langle A_l x, x' \rangle)_{l=1}^m).$$

Then, denoting the horizontal Brownian motion by $\mathbb{B}(t)$ we have

$$\mathbb{B}(t) = \left(B(t), \underbrace{\int_0^t \langle A_1 B(t), dB(t) \rangle}_{\text{Lévy area}}, \dots, \int_0^t \langle A_m B(t), dB(t) \rangle \right) \quad \forall t \geqslant 0.$$

where $(B_1(t), \dots, B_n(t))$ is the standard Brownian motion on $\mathbb{R}^n$.

Observation: The projection of $\mathbb{B}$ onto the horizontal variables coincides with the standard Brownian motion.

For $(x, v) \in \mathbb{G}$, let $\Pi : C_0(\mathbb{R}^n) \to C_0(\mathbb{G})$ be defined as

$$\Pi f(x, v) = f(x).$$

When $\mathbf{Q}_t$ is the horizontal heat semigroup on $C_0(\mathbb{G})$, $\mathbf{Q}_t \Pi f = \Pi Q_t f$, where Q_t is the classical heat semigroup on $\mathbb{R}^n$.

We can also consider the full Laplacian on $\mathbb{G}$ by adding the second order differential operators from $\mathfrak{g}_2$. Let us define

$$\mathbf{\Delta} = \mathbf{\Delta}_h + \mathbf{\Delta}_v, \quad \mathbf{\Delta}_v = \sum_{j=1}^{m} \frac{\partial^2}{\partial v_j^2}.$$

$\Delta_{\mathbb{G}}$ is also left-translation invariant and it generates a Markov semigroup on $\mathrm{L}^2(\mathbb{G})$. Moreover,

$$\mathbf{\Delta}\Pi = \mathbf{\Delta}_h\Pi + \mathbf{\Delta}_v\Pi = \Pi\Delta \quad (\mathbf{\Delta}_v\Pi = 0)$$

its projection onto the horizontal space coincides with the standard Brownian motion.

What is the class of left-translation invariant Markov semigroups on $\mathbb{G}$ whose horizontal projections coincide with the standard Brownian motion?

Theorem

Let $\widetilde{\mathbf{Q}}_t$ be a left-translation invariant Markov semigroup on $\mathbb{G}$ such that $\widetilde{\mathbf{Q}}_t \Pi = \Pi Q_t$ on $C_0(\mathbb{G})$. Then, $\widetilde{\mathbf{Q}}_t$ is generated by

$$\mathbf{\Delta}_L = \mathbf{\Delta}_h + L_v$$

for some m-dimensional Lévy generator L in the direction of the vertical variables of $\mathbb{G}$.

$$\begin{aligned} Lf(w) &= -cf(w) + \mathrm{tr}(\Sigma \nabla^2 f(w)) + \langle b, \nabla f(w) \rangle \\ &\quad + \int_{\mathbb{R}^m} \left(f(w + w') - f(w) - \mathbb{1}_{\{|w'| \leqslant 1\}} \langle w', \nabla f(w) \rangle \right) \kappa(dw'), \quad f \in C_c^\infty(\mathbb{R}^m) \end{aligned}$$

where $c \geqslant 0$, Σ is a nonnegative definite matrix, $b \in \mathbb{R}^m$, and κ is a Lévy measure on $\mathbb{R}^m$ satisfying

$$\int_{\mathbb{R}^m} (1 \wedge |w|^2) \kappa(dw) < \infty.$$

$\mathbf{\Delta}_L$ is not necessarily self-adjoint but it is a normal operator on $\mathrm{L}^2(\mathbb{G})$.

Recall that any Lévy generator can be uniquely associated to a Lévy-Khintchine exponent $\psi : \mathbb{R} \to \mathbb{C}$ defined by

$$\psi(\lambda) = c - \langle \Sigma\lambda, \lambda \rangle + \mathrm{i}\langle b, \lambda \rangle + \int_{\mathbb{R}^m_*} (e^{\mathrm{i}\langle w, \lambda \rangle} - 1 - \mathrm{i}\langle w, \lambda \rangle \mathbb{1}_{\{|w| \leqslant 1\}})\kappa(dw).$$

Theorem

$\mathbf{\Delta}_L$ has purely continuous spectrum. Moreover, $\mathrm{Im}(\sigma(\mathbf{\Delta}_L)) = \mathrm{Range}(\mathrm{Im}(\psi))$. If ψ is real valued then $\sigma(\mathbf{\Delta}_L) = (-\infty, \psi(0)]$.

Thus, the spectrum of $\mathbf{\Delta}_L$ depends on L, as expected.

Outline

The Ornstein-Uhlenbeck semigroup on $\mathbb{G}$

Define

$$\mathbf{P}_t = \delta_{e^{-t}} \mathbf{Q}_{\frac{1-e^{-2t}}{2}}.$$

Since

$$\delta_c \mathbf{Q}_{tc^2} = \mathbf{Q}_t \delta_c \quad \text{for all } c > 0,$$

$\mathbf{P}_t$ is a Markov semigroup. It is called the OU semigroup on $\mathbb{G}$.

The generator of $\mathbf{P}_t$ is

$$\mathbf{A} = \mathbf{\Delta}_h - \mathbf{D},$$

where $\mathbf{D}$ is the generator of the dilation group $(\delta_{e^{-t}})_{t \geqslant 0}$ on $\mathbb{G}$.

$\mathbf{P}_t$ is ergodic with invariant distribution $\mathbf{p} = \mathbf{q}_{\frac{1}{2}}$, where $\mathbf{q}_{\frac{1}{2}}$ is the horizontal heat kernel at $t = \frac{1}{2}$.

Theorem (Lust-Piquard, 2009)

$\mathbf{P}_t$ *is non self-adjoint in* $\mathrm{L}^2(\mathbb{G}, \mathbf{p})$. *For any* $t > 0$,

$$\sigma(\mathbf{P}_t) \setminus \{0\} = \sigma_p(\mathbf{P}_t) = e^{-t\mathbb{N}_0}.$$

The key idea in Lust-Piquard's proof uses the scaling property of $\mathbf{\Delta}_h$.

For a Lévy generator L, define the perturbed operator

$$\mathbf{A}_L = \mathbf{\Delta}_L - \mathbf{D} = \mathbf{A} + L_v$$

Note that $\mathbf{\Delta}_L$ does not satisfy any scaling property in general.

Theorem

1. $\mathbf{A}_L$ *is the generator of a* Markov *semigroup* $\mathbf{P}^L_t$.
2. *If the Lévy measure has* finite log-moment, $\mathbf{P}^L$ *is ergodic. The invariant distribution of* $\mathbf{P}^L$*, denoted by* $\mathbf{p}_L$ *can be computed explicitly.*

We consider the semigroup $\mathbf{P}^L$ in the weighted space $\mathrm{L}^2(\mathbb{G}, \mathbf{p}_L)$.

Let ψ be the Lévy-Khintchine exponent of L and define

$$q_L(dv) = \int_{\mathbb{R}^m} e^{-\mathrm{i}\langle v,\lambda\rangle} \exp\left(\int_0^\infty \psi(-e^{-2s}\lambda)ds\right) d\lambda.$$

The above quantity makes sense due the finite log-moment condition mentioned before.

Also define $\Lambda_L : B_b(\mathbb{G}) \to B_b(\mathbb{G})$

$$\Lambda_L \mathrm{f}(x,v) = \int_{\mathbb{R}^m} \mathrm{f}(x, v+v')q_L(dv')$$

Theorem

For all $t \geqslant 0$,

$$\mathbf{P}_t\Lambda_L = \Lambda_L\mathbf{P}_t^L \quad \textit{on } \mathrm{L}^q(\mathbb{G},\mathbf{p}_L), q \geqslant 1$$

Theorem

1. *If the Lévy measure κ satisfies*

$$\int_{\mathbb{R}^m} e^{\epsilon|v|}\kappa(dv) < \infty \quad \textit{for some } \epsilon > 0,$$

then $\mathbf{P}_t^L$ is a compact operator on $\mathrm{L}^q(\mathbb{G}, \mathbf{p}_L)$ for any $q > 1$.

2. *Under the above condition, the $\mathrm{L}^q(\mathbb{G}, \mathbf{p}_L)$-spectrum is given by*

$$\sigma(\mathbf{P}_t^L) \setminus \{0\} = \sigma_p(\mathbf{P}_t^L) = e^{-t\mathbb{N}_0}.$$

The eigenspace of e^{-tn} is

$$\Lambda_L^{-1} e^{-\frac{\mathbf{\Delta}_h}{2}}(\mathcal{P}_n),$$

where $\mathcal{P}_n$ is the space of δ-homogeneous polynomials of degree n.

So, the spectrum of P_t^L does not depend on L !.

Let $\mathbb{H}$ be the Heisenberg group of dimension 3.

Theorem (Baudoin, Bonnefont, Chen '21)

Consider the OU semigroup $\mathbf{P}_t$ on $\mathrm{L}^2(\mathbb{H}, \mathbf{p})$ corresponding to the sub-Laplacian $\boldsymbol{\Delta}_h$. Then, for any $\epsilon > 0$,

$$\|\mathbf{P}_t f - \int_{\mathbb{G}} f d\mathbf{p}\|_{\mathrm{L}^2(\mathbb{G},\mathbf{p})} \leqslant C_\epsilon e^{-(1-\epsilon)t} \|f - \int_{\mathbb{G}} f d\mathbf{p}\|_{\mathrm{L}^2(\mathbb{G},\mathbf{p})},$$

where $C_\epsilon \to \infty$ as $\epsilon \downarrow 0$.

- Consider the OU semigroup on $\mathbb{G}$ corresponding to

$$\mathbf{A}_{\mathbb{G}} = \boldsymbol{\Delta} - \mathbf{D} \quad (\text{non self-adjoint})$$

It has spectral gap of size 1. Can we prove Poincaré inequality using the spectral gap?

- If the answer to the first question is 'yes', can we use it to improve the result of Baudoin-Bonnefont-Chen for any Carnot group of step 2?

Thank You for your attention!